§16.? Surface Integrals

Last time: $\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v),y(u,v),z(u,v))\,|\vec{S}_u \times \vec{S}_v|\,dA$

where $\vec{S}(u,v)$ parameterizes the surface S on domain D

Ex Compute $\iint_S x^2\,dS$ for S the surface of the unit sphere centered at the origin

Sol: We parameterize S via

derived from spherical coords

$$S(\theta,\varphi) = \langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi)\rangle$$
on $(\theta,\varphi) \in [0,2\pi]\times[0,\pi]$

$\vec{S}_\theta = \langle -\sin(\varphi)\sin(\theta), \sin(\varphi)\cos(\theta), 0\rangle$

$\vec{S}_\varphi = \langle \cos(\varphi)\cos(\theta), \cos(\varphi)\sin(\theta), -\sin(\varphi)\rangle$

$$\vec{S}_\theta \times \vec{S}_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(\varphi)\sin(\theta) & \sin(\varphi)\cos(\theta) & 0 \\ \cos(\varphi)\cos(\theta) & \cos(\varphi)\cos(\theta) & -\sin(\varphi) \end{vmatrix}$$

$$= \langle -\sin^2(\varphi)\cos(\theta), -(\sin^2(\varphi)\sin(\theta)), -\sin\varphi\cos\varphi\sin^2\theta - \sin(\varphi)\cos(\varphi)\cos^2(\theta)\rangle$$

$$= -\sin(\varphi)\langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi)\rangle$$

$$|\vec{S}_\theta \times \vec{S}_u| = \sin(\varphi)\sqrt{\sin^2(\varphi)\cos^2(\theta) + \sin^2(\varphi)\sin^2(\theta) + \cos^2(\varphi)}$$
$$= \sin(\varphi)\sqrt{1}$$
$$= \sin(\varphi)$$

$$\therefore \iint_S x^2\,dS = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} \underbrace{\sin^2(\varphi)\cos^2(\theta)}_{x^2} \cdot \sin(\varphi)\, d\varphi\, d\theta$$

Fubini's $\rightarrow$ $= \int_0^{2\pi} \cos^2(\theta)\, d\theta \cdot \int_0^{\pi} \sin^3(\varphi)\, d\varphi$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi} (1+\cos(2\theta))\,d\theta \cdot \int_{\varphi=0}^{2\pi} \sin(\varphi)(1-\cos^2(\varphi))\,d\varphi \qquad \begin{array}{l} u=\cos(\varphi) \\ du=-\sin(\varphi)d\varphi \end{array}$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{2\pi} \cdot \int_0^{\pi} -(1-u^2)\,du$$

$$= \frac{1}{2}(2\pi - 0) \cdot \left(-\left[u - u^3/3\right]_0^{\pi}\right)$$

$$= \pi\left(-\left(\left(-1-\frac{1}{3}(-1)\right) - \left(1-\frac{1}{3}\right)\right)\right) = \boxed{\frac{4\pi}{3}}$$

WANT: A theory of surface integrals of vector fields...
First we need to understand what "orientation" should mean for surfaces
(changing orientation negates integrals)

Orientation $\approx$ choice of direction
$\uparrow$ orientation should be controlled by the normal vector of the tangent line to the surface at a given point
$\rightarrow$ $\vec{S}_u \times \vec{S}_v$ should point "out" or "up" for positive orientation

Ex:

Pointing outward = pos. orientation

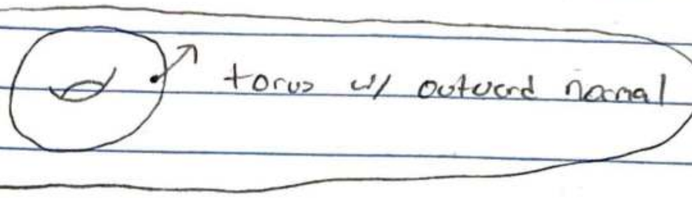
$\updownarrow$ @ every point

Pointing inward = neg. orientation

Q: Can we always use a consistent orientation for a surface?

Möbius Band → Surface = cylinder w/ half twist

→ non-orientable i.e. it has no consistent choice of normal



NB! Our surface integral from here on out will assume an orientable surface i.e. $\vec{n}(u,v) = \dfrac{\vec{s}_u \times \vec{s}_v}{|\vec{s}_u \times \vec{s}_v|}$ for parameterization

$\vec{s}(u,v)$ is consistent

→ e.g. the Möbius band is excluded

Def: Given a vector field $\vec{v}$ on $\mathbb{R}^3$ and an orientable surface S with parameterization $\vec{s}(u,v)$, the flux of $\vec{v}$ across S is

$$\iint_S \vec{v} \cdot d\vec{s} = \iint_S \vec{v} \cdot \vec{n}(u,v)\, dS$$

$$= \iint_S \vec{v} \cdot \frac{\vec{s}_u \times \vec{s}_v}{|\vec{s}_u \times \vec{s}_v|}\, dS$$

$$= \iint_D \vec{v} \cdot (\vec{s}_u \times \vec{s}_v)\, dA$$

Ex: Compute the flux of $\vec{v} = \langle z, y, x \rangle$ across the unit sphere centered at the origin

NB: when no orientation is given, assume the "counter-clockwise from above" or "outward" orientation

Sol: From earlier, S is parameterized by
$\vec{S}(\theta, \varphi) = \langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi) \rangle$ on $(\theta, \varphi) = [0, 2\pi] \times [0, \pi]$
and now

$\vec{S}_\theta \times \vec{S}_\varphi = -\sin(\varphi)\langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi) \rangle$

Q: Is that outward normal?
- Check "east pole" $(1,0,0)$
i.e. $(\theta, \varphi) = (0, \frac{\pi}{2})$

inward orientation

$(\vec{S}_\theta \times \vec{S}_\varphi)(0, \frac{\pi}{2}) = -1\langle 1, 0, 0 \rangle = \langle -1, 0, 0 \rangle$

Points inwards

$\therefore$ we must work w/ $-\vec{S}_\theta \times S_\varphi$ instead

$\therefore$ The flux of $\vec{v}$ across S is

$$\iint_S \vec{v} \cdot d\vec{S} = \iint_D \vec{v}(\varphi, \theta) \cdot (\vec{S}_u \times S_v)\, dA$$

$$= \int\int_D \langle \cos(\varphi), \sin(\varphi)\sin(\theta), \sin(\varphi)\cos(\theta) \rangle \cdot -(-\sin(\varphi)\langle \sin(\varphi)\cos(\theta), \sin(\varphi)\cos(\theta), \cos(\varphi) \rangle)\, dA$$

$$= \iint_D \sin(\varphi)\left(2\cos(\varphi)\sin(\varphi)\cos(\theta) + \sin^2(\varphi)\sin^2(\theta)\right) dA$$

$$= 2\iint_D \cos(\varphi)\sin^2(\varphi)\cos(\theta)\, dA + \iint_D \sin^3(\varphi)\sin^2(\theta)\, dA$$

$$= 2\iint_D \cos(\varphi)\sin^2(\varphi)\cos\theta\, dA = 2\int_0^{\pi}\int_0^{2\pi} \cos(\varphi)\sin^2(\varphi)\cos(\theta)\, d\theta\, d\varphi$$

$$= 2\int_0^{\pi} \cos(\varphi)\sin^2(\varphi)\left[\sin(\theta)\right]_0^{2\pi} d\varphi$$

$$= 2\int_0^{\pi} 0 \, d\varphi = 0$$

and $\iint_D \sin^3(\varphi)\sin^2(\theta)\, dA$

$$= \int_0^{\pi}\int_0^{2\pi} \sin^3(\varphi)\cdot \tfrac{1}{2}(1-\cos(2\theta))\, d\theta\, d\varphi$$

$$= \int_0^{\pi} \tfrac{1}{2}\sin^3(\varphi)\left[\theta - \tfrac{1}{2}\sin(2\theta)\right]_0^{2\pi} d\varphi$$

$$= \int_0^{\pi} \tfrac{1}{2}(2\pi - 0)\sin(\varphi)(1-\cos^2(\varphi))\, d\varphi$$

$u = \cos(\varphi)$ $\quad u(\pi) = -1$

$du = -\sin(\varphi)d\varphi$ $\quad u(0) = 1$

$$= \pi\int_{u=1}^{-1} -(1-u^2)\, du$$

$$= -\pi\left[u - \tfrac{1}{3}u^3\right]_1^{-1} = -\pi\left(\left(-1+\tfrac{1}{3}\right)-\left(1-\tfrac{1}{3}\right)\right) = \frac{4\pi}{3}$$

$\therefore$ The flux of $\vec{v}$ across S is

$$\iint_S \vec{v}\cdot d\vec{S} = 0 + \frac{4\pi}{3} = \boxed{\frac{4\pi}{3}}$$

Try @ Home!!

Compute the flux of $\vec{v}$ across S for $\vec{v} = \langle y, x, z\rangle$ on boundary of the solid enclosed by the paraboloid $z = 1 - x^2 - y^2$ and $z = 0$.